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1983 J. Phys. A: Math. Gen. 16 4177

(http://iopscience.iop.org/0305-4470/16/18/018)

J. Phys. A: Math. Gen. **16** (1983) 4177–4182. Printed in Great Britain

A note on the Atiyah–Singer index theorem

Luis Alvarez-Gaumé†

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

Received 22 July 1983

Abstract. A new proof of the Atiyah–Singer index theorem for the Dirac equation in the presence of external gauge and gravitational fields is presented.

1. Introduction

In the last few years there has been a growing interest in supersymmetry and its connections with global results in differential geometry and topology (Zumino 1977, Witten 1982a, b). Recently, a new proof of the Atiyah–Singer index theorem (Atiyah and Singer 1968a, b, 1971a, b, Atiyah and Segal 1968) based on supersymmetry was introduced (Alvarez-Gaumé 1983) and improved mathematically in Getzler (1983). In this note, we complete the work started in Alvarez-Gaumé (1983) by directly computing the Atiyah–Singer index density for the Dirac equation defined on an even-dimensional compact manifold M in the presence of an external gauge group G.

The reason why supersymmetry is naturally related to the Atiyah–Singer index theorem is as follows. Let us consider a supersymmetric $(0+1)$-dimensional field theory (i.e. supersymmetric quantum mechanics). This theory has N conserved charges Q_i, $i=1,\ldots,N$, which anticommute with the fermion number operator $(-1)^{\mathrm{F}}$, and which satisfy the supersymmetry algebra

$$\{Q_i, Q_j^*\}=2\delta_{ij}H, \qquad \{Q_i,(-1)^{\mathrm{F}}\}=0, \qquad \{\mathrm{Q_i}, \mathrm{Q}_j\}=0, \qquad i,j=1,\ldots,N, \tag{1}$$

where H is the Hamiltonian of our $(0+1)$-dimensional field theory. let Q, Q^* be any of the N supersymmetric charges Q_i, $i=1,\ldots,N$. Then the operator $\sqrt{2}S=Q+Q^*$ is Hermitian and satisfies $S^2=H$. Given an arbitrary eigenstate $|E\rangle$ of H, $H|E\rangle=E|E\rangle$, $E\neq 0$, $S|E\rangle$ is another state with the same energy. Therefore if $|E\rangle$ is a bosonic (fermionic) state, $S|E\rangle$ will be fermionic (bosonic), so that the non-zero energy states in the spectrum appear in Fermi–Bose pairs. Thus, the quantity $\mathrm{Tr}(-1)^{\mathrm{F}}\,\mathrm{e}^{-\beta H}$ (Witten 1982c), will only receive contributions from the zero energy states, and can be shown to be a topological invariant of the quantum theory (Witten 1982c). Since the bosonic zero energy states are determined by the solutions of the equation $Q|B\rangle=0$, and the fermionic zero modes are given by $Q^*|F\rangle=0$, it follows that $\mathrm{Index}(Q)=\mathrm{Kernel}(Q)-\mathrm{Kernel}\,(Q^*)=\mathrm{Tr}(-1)^{\mathrm{F}}\,\mathrm{e}^{-\beta H}$, and we can calculate the index of Q if we know how to evaluate $\mathrm{Tr}(-1)^{\mathrm{F}}\,\mathrm{e}^{-\beta H}$ in the $\beta\to 0$ limit (high temperature). In order to compute the trace we use the fact that it has a functional integral represention (Cecotti and Girardello 1982), which is exactly the same as the functional integral representation for the

† Research is supported in part by the National Science Foundation under Grant No PHY82-15249, and by the Harvard Society of Fellows.

partition function but with the fermions integrated over with periodic boundary conditions:

$$\mathrm{Tr}(-1)^{\mathrm{F}}\,\mathrm{e}^{-\beta H}=\int_{\mathrm{PBC}} \mathrm{d}\phi(t)\,\mathrm{d}\psi(t)\exp-\int_0^\beta L_{\mathrm{E}}(t)\,\mathrm{d}t, \tag{2}$$

$L_{\mathrm{E}}(t)$ is the Euclidean Lagrangian defining the $(0+1)$-dimensional field theory, $\phi(t)$, $\psi(t)$ are the bosonic and fermionic fields in the theory, and PBC is just a shorthand notation to indicate that both $\phi(t)$, $\psi(t)$ are integrated over with periodic boundary conditions in β, i.e. $\phi(\beta)=\phi(0)$, $\psi(\beta)=\psi(0)$.

In Alvarez-Gaumé (1983) it was shown that the index theorem for all the classical complexes (DeRahm, Dirac, Hirzebruch and Dolbeault complexes) can be obtained using supersymmetry in the way just described, and the Lagrangian one has to use follows from the supersymmetric nonlinear σ-model (de Vecchia and Ferrara 1977, Witten 1977, Freedman and Townsend 1981) that one obtains by dimensionally reducing from $(1+1)$ to $(0+1)$ dimensions:

$$L=\tfrac{1}{2}g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j+\tfrac{1}{2}\mathrm{i}g_{ij}\psi^i_\alpha[(\mathrm{d}/\mathrm{d}t)\psi^j_\alpha+\Gamma^j_{kl}\dot{\phi}^k\psi^l_\alpha]+\tfrac{1}{4}R_{ijkl}\psi^i_1\psi^j_1\psi^k_2\psi^l_2,$$
$$i,j=1,n, \qquad \alpha,\beta=1,2. \tag{3}$$

$g_{ij}(\phi)$ is the metric on the manifold M, Γ^i_{jk} is the Christoffel connection and R_{ijkl} is the curvature tensor, $\psi^i_\alpha(t)$, $\alpha=1,2$, are real anticommuting fermi fields.

From the above, it follows that the first step needed in the derivation of the index theorem for the Dirac equation in the presence of a gauge field is to find a $(0+1)$-dimensional Lagrangian whose Hamiltonian is the square of the Dirac operator of interest. This is done in § 2. Section 3 contains the derivation of the index theorem, and § 4 presents the conclusions.

2. The Dirac equation

The Lagrangian defined by equation (3) is invariant under a supersymmetry transformation which involves two constant anticommuting real Grassmann numbers ε_1, ε_2 (Freedman and Townsend 1981). Let us now impose the constraint $\psi^i_1=\psi^i_2=\psi^i/\sqrt{2}$; (3) becomes:

$$L=\tfrac{1}{2}g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j+\tfrac{1}{2}\mathrm{i}g_{ij}(\phi)\psi^i(\mathrm{d}\psi^j/\mathrm{d}t+\Gamma^j_{kl}\dot{\phi}^k\psi^l) \tag{4}$$

with a single supersymmetry corresponding to $\varepsilon_1=-\varepsilon_2=\varepsilon$, and the supersymmetry current is $Q=g_{ij}(\phi)\psi^i\dot{\phi}^j$. Introducing a vierbein frame $e^a_i(\phi)$ such that $g_{ij}=e^a_ie^b_j\delta_{ab}$, $E^i_ae^a_j=\delta^i_j$, and redefining the fermion fields: $\psi^a=e^a_i\psi^i$, (4) becomes

$$L=\tfrac{1}{2}g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j+\tfrac{1}{2}\mathrm{i}\delta_{ab}\psi^a(\mathrm{d}\psi^b/\mathrm{d}t+\dot{\phi}^i\omega^a_ib\psi^b),$$
$$\omega^a_ib=-E^k_b(\partial_ie^a_k-\Gamma^l_{ik}e^a_l), \tag{5}$$

where ω^a_ib is the spin connection. Note that (5) is invariant under local SO(n) rotations: $\psi^a\to L^ab(\phi)\psi^b$, $\omega^a_ib\to L^a_c\omega^c_idL^db+L^ac\partial_iL^cb$, $\delta_{ab}L^acL^bd=\delta_{cd}$. If we canonically quantise (5), then $\{\psi^a,\psi^b\}=\delta^{ab}$, and the supercharge becomes $Q=\mathrm{i}\gamma^aD_a/\sqrt{2}$, so that the Hamiltonian is given by

$$H=\tfrac{1}{2}(\mathrm{i}\gamma^aD_a)^2, \qquad D_a=E^i_a(\partial_i+\tfrac{1}{2}\omega_{iab}\sigma^{ab}), \qquad \sigma^{ab}=\tfrac{1}{4}[\gamma^a,\gamma^b]. \tag{6}$$

D_a is the covariant derivative acting on spinors, and the γ^a's are the usual Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$. In this form of the theory, the fermion number operator $(-1)^F$ is simply γ_5. (Actually γ_{d+1}, $d = \dim M$.)

Since we are interested in the Dirac equation in the presence of gauge interactions, let G be a gauge group acting on M with gauge connection $A_i^\alpha(\phi)$, and gauge curvature $F_{ij}^\alpha(\phi) = \partial_i A_j^\alpha - \partial_j A_i^\alpha + g f^{\alpha\beta\gamma} A_i^\beta A_j^\gamma$ (g is the gauge coupling constant), $\alpha\beta = 1, \dots, \dim G$. If the spinors on which the Dirac operator acts transform according to the representation of G generated by $(T^\alpha)_{A,B}$, $A,B = 1, \dots, \dim T$, the eigenvalue problem for the Dirac equation can be written:

$$\mathrm{i}\gamma^i(\partial_i + \tfrac{1}{2}\omega_{iab}\sigma^{ab} + \mathrm{i}gA_i^\alpha T^\alpha)_{AB}\psi\lambda_B = \lambda(\psi_\lambda)_A, \qquad T^{\alpha\dagger} = T^\alpha, \tag{7}$$

with only group indices explicitly indicated. In order to find the one-dimensional analogue of (7), we introduce for each index A, $A = 1, \dots, \dim T$, a pair of fermionic creation and annihilation operators: c_A^*, c_A, such that

$$\{c_A, c_B\} = 0, \qquad \{c_A^*, c_B\} = \delta_{AB}. \tag{8}$$

In the Hilbert space generated by the c's we can consider states of the form

$$|\psi\rangle = \sum_A \psi_A(\phi) c_A^* |0\rangle \tag{9}$$

(spinor indices being omitted). Then (7) can be recast as

$$\mathrm{i}\gamma^i(\partial_i + \tfrac{1}{2}\omega_{iab}\sigma^{ab} + \mathrm{i}gA_i^\alpha c^* T^\alpha c)|\psi\rangle = \lambda|\psi\rangle. \tag{10}$$

A trivial feature of (7) and (10) which will be useful to us later is that if $|\psi, \lambda\rangle$ is an eigenfunction of the Dirac equation with eigenvalue $\lambda \neq 0$, then $\gamma_5|\psi, \lambda\rangle$ is also an eigenfunction but with opposite eigenvalue. Now it is easy to generalise (5) so as to include the gauge field. Consider the Lagrangian

$$\begin{aligned} L = {} & \tfrac{1}{2}g_{ij}(\phi)\dot\phi^i\dot\phi^j + \tfrac{1}{2}\mathrm{i}\delta_{ab}\psi^a(\mathrm{d}\psi^b/\mathrm{d}t + \omega_{iab}\dot\phi^i\psi^b) \\ & + \mathrm{i}c_A^*(\mathrm{d}c_A/\mathrm{d}t - \mathrm{i}gA_i^\alpha(\phi)\dot\phi^i(T^\alpha)_{AB}c_B) - \tfrac{1}{2}\mathrm{i}\psi^a\psi^b g F_{ab}^\alpha c_A^* T_{AB}^\alpha c_B. \end{aligned} \tag{11}$$

The Hamiltonian generated by (11) is just the square of the operator appearing in the left-hand side of (7)†.

Even though the theory defined by (11) has the desired Hamiltonian. Namely $\frac{1}{2}(\mathrm{i}\not{D})^2$, the theory is clearly not supersymmetric because there is a mismatch between the bosonic and fermionic degrees of freedom. Thus the arguments presented in the introduction to relate (2) and the index of the Dirac equation do not immediately apply. The desired result follows nonetheless because $(\mathrm{i}\not{D})^2$ commutes with γ_5, and as mentioned before, the non-zero eigenvalues of the Dirac equation come in pairs $(\lambda, -\lambda)$, so that

$$\mathrm{Tr}\, \gamma_5\, \mathrm{e}^{-\beta(\mathrm{i}\not{D})^2} = n^{E=0}(\gamma_5 = +1) - n^{E=0}(\gamma_5 = -1). \tag{12}$$

$n^{E=0}(\gamma_5 = \pm 1)$ stands for the number of zero eigenvalues of $(\mathrm{i}\not{D})^2$ with $\gamma_5 = \pm 1$. Thus, computing the index of the Dirac operator is equivalent to the evaluation of (2) for the Lagrangian given in (11), with the condition that the functional integral be restricted

† There is a subtlety in equation (11) related to the operator ordering chosen. When the gauge field is absent, the most natural prescription is to choose the operator ordering which guarantees that $H = Q^2$ after canonical quantisation. Once this ordering is chosen for the purely geometrical part, there is no further ambiguity in (11).

to the space of one particle states for the (c^*, c) fermions. This condition is necessary since we want to compute the index of $\mathrm{i}\not{D}$ only in the representation T^α of the gauge group G, and not in any of its tensor products. Since only the one-particle states of the c-fermions carry the representation T^α, we must impose this constraint on the functional integral. We now proceed to calculate explicitly the index density for $\mathrm{i}\not{D}$.

3. The index density

We showed in § 2 that the index of the Dirac operator is given by (2), where $L_{\mathrm{E}}(t)$ is the Euclidean version of (11). In order to obtain the characteristic polynomial (or index density) for the Dirac operators, we only need to evaluate (12) in the $\beta \to 0$ limit. The functional integral representation of (12) is exactly as in (2) as far as ϕ and ψ are concerned, while the c's and c^*'s are integrated over with antiperiodic boundary conditions. This follows from the fact that the trace has to be defined over one particle fermionic states for the c's.

In the $\beta \to 0$ limit, the functional integral is dominated by time-independent constant configurations, i.e., $\phi^i(t) = \phi_0^i$, $\psi^i(t) = \psi_0^i$, $c_A = c_A^* = 0$, and the leading small-β behaviour is just given by the second-order terms in the expansion of $L(t)$ around (ϕ_0, ψ_0). The expansion of $\phi^i(t)$, $\psi^i(t)$ around the constant configurations is simplified if it is carried out using normal coordinates. After some algebra, the second-order term in the expansion of (11) is (Alvarez-Gaumé *et al* 1981, Alvarez-Gaumé 1983):

$$L = \tfrac{1}{2} g_{ij}(\phi_0)\dot{\xi}^i\dot{\xi}^j + \tfrac{1}{4}\mathrm{i}R_{ijab}(\phi_0)\psi_0^a\psi_0^b\xi^i\dot{\xi}^j + \tfrac{1}{2}\mathrm{i}\delta_{ab}\eta^a \mathrm{d}\eta^b/\mathrm{d}t$$
$$+ \mathrm{i}c_A^* \mathrm{d}c_A/\mathrm{d}t - \tfrac{1}{2}\mathrm{i}g\psi_0^a\psi_0^b F_{ab}^\alpha(\phi_0)c_A^* T_{AB}c_B. \tag{13}$$

ξ^i and η^a are the fluctuations of ϕ^i and ψ^a around (ϕ_0^i, ψ_0^a) and they are supposed to be non-constant in order to avoid overcounting. In this way, the functional integral splits nicely between constant and non-constant configurations. Notice also that we need not expand the c's because (11) is already second order in small fluctuations with respect to c-fermions. In terms of (13), the trace (12) decomposes into two factors: one is the partition function for a set of bosonic oscillators (the first two terms in (13)), and the other is the trace over one particle states of $\mathrm{e}^{-\beta H'}$, where H' describes the Hamiltonian for a set of fermionic oscillators (the last two terms in (13)). The trace is normalised by dividing by the same trace with gauge and gravitational fields omitted. The third term in (13) does not contribute. Since the manifold has dimension $2n$, we have to include a factor of $(2\pi)^{-n}$ coming from the usual Feynman measure for the constant modes, and a factor of i^n since we are also integrating over constant real fermionic configurations: ψ_0^i, $\psi_o^{i*} = \psi_0^i$. The result of this computation is

$$\operatorname{ind}(\mathrm{i}\not{D}) = \frac{\mathrm{i}^n}{(2\pi)^n}\int \mathrm{d\,vol}\int (\mathrm{d}\psi_0)(\operatorname{Tr}\exp(-\tfrac{1}{2}\mathrm{i}\psi_0^a\psi_0^b F_{ab}^\alpha T^\alpha))\prod_{l=1}^{n}\frac{\mathrm{i}x_l/2}{\sinh(\mathrm{i}x_l/2)} \tag{14}$$

where the x_l's are the skew eigenvalues of the matrix $\frac{1}{2}R_{abcd}\psi_0^c\psi_0^d$, and the index density is obtained by expanding the integrand of (14) to $2n$th order in the ψ_0's. Any other terms in the expansion are irrelevant due to the presence of the Grassmann integration over the ψ_0's.

In a more geometrical language, the ψ_0's play the same role as the basis of one-forms on the manifold: $\mathrm{e}^a = \mathrm{e}_i^a(\phi)\mathrm{d}\phi^i$. Then in terms of the curvature and gauge field strength two-forms

$$R_{ab}=\tfrac{1}{2}R_{abcd}e^c\wedge e^d, \qquad F=\tfrac{1}{2}(gF^{\alpha}_{ab}T^{\alpha})e^a\wedge e^b, \tag{15a, b}$$

we can form the following two polynomials

$$\mathrm{ch}(F)=\mathrm{Tr}\ \mathrm{e}^{F/2\pi} \tag{16a}$$

$$\hat{A}(M)=\prod_{l}(\omega_\alpha/4\pi)/\sinh(\omega_\alpha/4\pi). \tag{16b}$$

In (16*a*) the trace runs over the relevant representation of G under consideration, and the ω_α's appearing in (16*b*) are the formal skew eigenvalues of the antisymmetric matrix of two-forms (15*a*). $\mathrm{ch}(F)$ is known in the mathematical literature as the Chern character of the principal bundle defined by the gauge field, and $\hat{A}(M)$ is known as the Dirac genus of the manifold M (see Eguchi *et al* (1980) for more details). In terms of (16*a*, *b*), we see that the result (14) for the index of $\mathrm{i}\not{D}$ can be rewritten in terms of (16*a*, *b*) as the term proportional to the volume form in the product $\mathrm{ch}(F)\hat{A}(M)$

$$\mathrm{ind}(\mathrm{i}\not{D})=\int_M(\mathrm{ch}(F)\hat{A}(M))_{\mathrm{vol}}. \tag{17}$$

This is the Atiyah–Singer index theorem for the Dirac equation on a compact manifold, including the contribution due to the presence of a gauge field. In particular, for a four-dimensional manifold

$$\mathrm{ind}(\mathrm{i}\not{D})=\frac{(\dim T)}{192\pi^2}\int\mathrm{Tr}\ R\wedge R+\frac{1}{8\pi^2}\int\mathrm{Tr}\ F\wedge F \tag{18}$$

as should be (see for instance Eguchi *et al* (1980).

4. Conclusions

We have shown that by using ideas inspired by supersymmetry, we can obtain the general form of the Atiyah–Singer index theorem for the Dirac equation. In fact, a judicious choice of the bundle F together with equation (17) allows us to derive easily the index theorem for all the classical complexes.

Another interesting aspect of the method presented here is that it allows a simplification of the computation of anomalies for axial vector currents (Adler 1969, Bell and Jackiw 1969, Gross and Jackiw 1972, for reviews of the anomalies see Adler 1970, Jackiw 1972). This computation ordinarily requires the evaluation of a trace of the form $\Sigma_n\,\psi_n^\dagger(x)L\psi_n(x)$, with L as an algebraic or differential operator, and where the ψ_n's are the eigenfunctions of the Dirac operator in the presence of external gravitational and/or gauge fields†. We have shown that these traces can easily be transformed into one-dimensional functional integrals, and thus, that the problem of computing anomalies is reduced to the somewhat simpler problem of computing partition functions in ordinary quantum mechanics.

† In the physics literature, there have been papers (Nielsen *et al* 1977, 1978, Jackiw *et al* 1978) where the local density for the Atiyah–Singer index theorem for the Dirac equation has been obtained in four dimensions, but using very different methods from ours.

Acknowledgments

This work was started while visting Princeton in the spring of 1983. I would like to thank the Princeton Theory Group for their warm hospitality. I would also like to thank P Ginsparg for useful remarks.

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